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著者	DRAGOMIR Sorin, TOMMASOLI Andrea
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ON p -HARMONIC MAPS INTO SPHERES

By

Sorin DRAGOMIR and Andrea TOMMASOLI

Abstract. We study the topology of p -harmonic maps $\phi : M \rightarrow S^v$ from a compact Riemannian manifold M into a sphere S^v . We also show that any p -energy minimizing map $\phi : M \rightarrow S^v$ omitting a totally geodesic submanifold of codimension two $\Sigma \subset S^v$ is of class C^1 . This extends results by B. Solomon, [13], from harmonic to p -harmonic maps.

1. Introduction

Let (M, g) and S be Riemannian manifolds and $2 \leq p < \infty$. Let $\iota : S \hookrightarrow \mathbf{R}^N$ be an isometric immersion in some Euclidean space \mathbf{R}^N and let $W^{1,p}(M, S)$ be the Sobolev space

$$W^{1,p}(M, S) = \{\phi \in W^{1,p}(M, \mathbf{R}^N) : \phi(x) \in S \text{ for a.e. } x \in S\}.$$

Let us consider the p -energy integral

$$E_p(\phi) = \int_M \|d\phi\|^p d \operatorname{vol}(g), \quad \phi \in W_{\operatorname{loc}}^{1,p}(M, S).$$

We study p -harmonic maps i.e. C^∞ maps $\phi : M \rightarrow N$ which are critical points of E_p (with respect to any variation of compact support). The theory of p -harmonic maps has undergone an ample development both from the point of view of partial differential equations (cf. e.g. [2]–[3], [5], [7]–[9] and [15]) and differential geometry (cf. e.g. [6], [10], [14] and [16]). When $S = S^v \subset \mathbf{R}^{v+1}$ is the standard sphere we show that any nonconstant p -harmonic map $\phi : M \rightarrow S^v$ either links or meets $\Sigma = \{x = (x_1, \dots, x_{v+1}) \in S^v : x_1 = x_2 = 0\}$ (cf. Section 2 for definitions). This generalizes a result by B. Solomon, [13], where the previous statement

was established for $p = 2$. A map $\phi \in W^{1,p}(M, S)$ is *p-energy minimizing* if $E_p(\phi) \leq E_p(\psi)$ for any compact set $K \subset M$ and any map $\psi \in W_{\text{loc}}^{1,p}(M, S)$ such that $\psi = \phi$ a.e. in $M \setminus K$. Building on the results of N. Nakauchi, [9], we show that any *p-energy minimizing* map $\phi \in W_{\text{loc}}^{1,p}(M, S^v)$ which omits a neighborhood of Σ in S^v is C^1 . Given a subset $E \subset M$ we set

$$H_\sigma(E) = \lim_{\varepsilon \rightarrow 0^+} \inf \left\{ \sum_{v=1}^{\infty} \text{diam}(X_v)^\sigma : \bigcup_{v=1}^{\infty} X_v \supset E, \text{diam}(X_v) \leq \varepsilon \right\}$$

where diameters are meant with respect to the Riemannian distance function on M . Then $h(E) = \inf\{\sigma \geq 0 : H_\sigma(E) = 0\}$ is the *Hausdorff dimension* of E . For each map $\phi : M \rightarrow S$ let $\text{Reg}(\phi)$ consist of all points $x \in M$ such that ϕ is continuous at x . Then $\text{Sing}(\phi) = M \setminus \text{Reg}(\phi)$ is the *singular set* of ϕ . We establish an upper bound on the Hausdorff dimension $h[\text{Sing}(\phi)]$ of a *p-energy minimizing p-harmonic* map $\phi \in W_{\text{loc}}^{1,p}(M, S)$ whose target manifold S is covered by a warped product manifold (cf. Theorem 3 below).

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2. p-Harmonic Maps into Warped Products

Let (L, g_L) be a Riemannian manifold and $S = L \times_w \mathbf{R}$ a warped product where $w \in C^\infty(S)$, $w > 0$, is the warping function. S carries the Riemannian metric $h = \Pi_1^* g_L + w^2 dt \otimes dt$ where $\Pi_1 : S \rightarrow L$ is the natural projection. Let M be a compact orientable n -dimensional Riemannian manifold. By a result of B. Solomon, [13], for any harmonic map $\phi : M \rightarrow S$ there is $t_\phi \in \mathbf{R}$ such that $\phi(M) \subset L \times \{t_\phi\}$. The purpose of this section is to establish a similar result for *p-harmonic* maps.

LEMMA 1. *Let $\phi : M \rightarrow L \times_w \mathbf{R}$ be a p-harmonic map ($p \geq 2$). Let $u = \Pi_2 \circ \phi$ where $\Pi_2 : S \rightarrow \mathbf{R}$ is the projection. Then u is a solution to the elliptic equation*

$$(1) \quad \text{div}[\|d\phi\|^{p-2}(w \circ \phi)^2 \nabla u] = \|d\phi\|^{p-2}(w \circ \phi)(w_t \circ \phi).$$

In particular if w is a function on L (i.e. $\partial w / \partial t = 0$) then $\phi(M) \subset L \times \{t_\phi\}$ for some $t_\phi \in \mathbf{R}$.

PROOF. Let $F = \Pi_1 \circ \phi$. Let $\varphi \in C^\infty(M)$ and let us set

$$\phi_s(x) = (F(x), u(x) + s\varphi(x)), \quad x \in M, |s| < \varepsilon,$$

so that $\{\phi_s\}_{|s|<\varepsilon}$ is a smooth 1-parameter variation of ϕ . Let g be the Riemannian metric on M . The Hilbert-Schmidt norm of the differential $d\phi_s$ is given by

$$(2) \quad \|d\phi_s\|^2 = \|dF\|^2 + (w \circ \phi_s)^2 \{ \|\nabla u\|^2 + 2sg(\nabla u, \nabla \varphi) + O(s^2) \}.$$

As $\phi : M \rightarrow S$ is p -harmonic $\{dE_p(\phi_s)/ds\}_{s=0} = 0$ hence

$$(3) \quad \int_M \|d\phi\|^{p-2} \{ (w \circ \phi)(w_t \circ \phi) \|\nabla u\|^2 + (w \circ \phi)^2 g(\nabla u, \nabla \varphi) \} d \text{vol}(g) = 0.$$

The identity

$$\|d\phi\|^{p-2} (w \circ \phi)^2 g(\nabla u, \nabla \varphi) = \text{div}[\varphi \|d\phi\|^{p-2} (w \circ \phi)^2 \nabla u] - \varphi \text{div}[(w \circ \phi)^2 \|d\phi\|^{p-2} \nabla u]$$

together with Green's lemma yields

$$\int_M \varphi \{ \|d\phi\|^{p-2} (w \circ \phi)(w_t \circ \phi) - \text{div}[\|d\phi\|^{p-2} (w \circ \phi)^2 \nabla u] \} d \text{vol}(g) = 0$$

hence (as $\varphi \in C^\infty(M)$ is arbitrary) the equation (1) holds. If $w_t = 0$ then (1) becomes $\text{div}[\|d\phi\|^{p-2} (w \circ \phi)^2 \nabla u] = 0$. Consequently

$$\text{div}[\|d\phi\|^{p-2} (w \circ \phi)^2 u \nabla u] = \|d\phi\|^{p-2} (w \circ \phi)^2 \|\nabla u\|^2$$

and integrating over M leads (by Green's lemma) to $\nabla u = 0$ hence u is constant. Q.e.d.

Let $S^v = \{x = (x_1, \dots, x_{v+1}) \in \mathbf{R}^{v+1} : x_1^2 + \dots + x_{v+1}^2 = 1\}$ and $\Sigma = \{x \in S^v : x_1 = x_2 = 0\}$. We recall that $S^v \setminus \Sigma$ is isometric to the warped product $S_+^{v-1} \times_f S^1$ where

$$S_+^{v-1} = \{y = (y', y_v) \in \mathbf{R}^v : y \in S^{v-1}, y_v > 0\},$$

$$f : S_+^{v-1} \times S^1 \rightarrow (0, +\infty), \quad f(y, \zeta) = y_v, \quad y \in S_+^{v-1}, \zeta \in S^1 \subset \mathbf{C}.$$

Indeed the map

$$I(y, \zeta) = (y_v u, y_v v, y'), \quad y \in S_+^{v-1}, \zeta = u + iv \in S^1,$$

is an isometry of $S_+^{v-1} \times_f S^1$ endowed with the Riemannian metric $\pi_1^* g_{v-1} + f^2 \pi_2^* g_1$ onto $(S^v \setminus \Sigma, g_v)$. Here $\pi_1 : S_+^{v-1} \times_f S^1 \rightarrow S_+^{v-1}$ and $\pi_2 : S_+^{v-1} \times_f S^1 \rightarrow S^1$ are the natural projections. Also g_N denotes the canonical Riemannian metric on

the sphere $S^N \subset \mathbf{R}^{N+1}$. The next section is devoted to p -harmonic maps from M into $S^v \setminus \Sigma$.

3. p -Harmonic Maps Omitting a Totally Geodesic Submanifold

Let $\Sigma \subset S^v$ be a codimension 2 totally geodesic submanifold. A continuous map $\phi : M \rightarrow S^v$ *meets* Σ if $\phi(M) \cap \Sigma \neq \emptyset$. Let $\phi : M \rightarrow S^v$ be a continuous map that doesn't meet Σ . Then ϕ is said to *link* Σ if the map $\phi : M \rightarrow S^v \setminus \Sigma$ is not null-homotopic.

THEOREM 1. *Let $\phi : M \rightarrow S^v$ be a nonconstant p -harmonic ($p \geq 2$) map. Then ϕ either links or meets Σ .*

For $p = 2$ this is B. Solomon's Theorem 1 in [13], p. 155. The proof of Theorem 1 is by contradiction. We wish to apply Lemma 1 above for $L = S_+^{v-1}$ and $w \in C^\infty(S_+^{v-1})$ given by $w(y) = y_v$. Let $\phi : M \rightarrow S^v \setminus \Sigma$ be a null-homotopic p -harmonic map. Let $E : \mathbf{R} \rightarrow S^1$ be the exponential map $E(t) = e^{it}$. We endow $S_+^{v-1} \times \mathbf{R}$ with the warped product metric $\Pi_1^* g_{v-1} + (w \circ \Pi_1)^2 dt \otimes dt$. Then $\pi = (\text{id}, E)$ is a local isometry of $S_+^{v-1} \times_w \mathbf{R}$ onto $S_+^{v-1} \times_f S^1$. There is a coordinate system on \mathbf{R}^{v+1} such that $\Sigma = \{x \in S^v : x_1 = x_2 = 0\}$. Let then $\tilde{\psi} = I^{-1} \circ \phi$ and let us set $F = \pi_1 \circ \tilde{\psi}$ and $\tilde{u} = \pi_2 \circ \tilde{\psi}$. Let $x_0 \in M$ and $\zeta_0 = \tilde{u}(x_0) \in S^1$. Let $t_0 \in \mathbf{R}$ such that $E(t_0) = \zeta_0$. As $\tilde{\psi} : M \rightarrow S_+^{v-1} \times S^1$ is null-homotopic it follows that $\tilde{u}_* \pi_1(M, x_0) = 0$ hence by standard homotopy theory (cf. e.g. Proposition 5.3 in [4], p. 43) there is a unique function $u : M \rightarrow \mathbf{R}$ such that $u(x_0) = t_0$ and $E \circ u = \tilde{u}$. As \tilde{u} is smooth it follows that $u \in C^\infty(M)$ as well. It is an elementary matter that

LEMMA 2. *Let S and \tilde{S} be Riemannian manifolds and $\pi : S \rightarrow \tilde{S}$ a local isometry. Let $\tilde{\phi} : M \rightarrow \tilde{S}$ be a p -harmonic map ($p \geq 2$) from a compact orientable Riemannian manifold M into \tilde{S} . Then any smooth map $\phi : M \rightarrow S$ such that $\pi \circ \phi = \tilde{\phi}$ is p -harmonic.*

By Lemma 2 it follows that $\psi = (F, u) : M \rightarrow S_+^{v-1} \times_w \mathbf{R}$ is a p -harmonic map. Then (by Lemma 1) $\psi(M) \subset S_+^{v-1} \times \{t_\psi\}$ for some $t_\psi \in \mathbf{R}$. To end the proof of Theorem 1 we establish

LEMMA 3. *Any p -harmonic ($p \geq 2$) map $\phi : M \rightarrow S_+^{v-1}$ of a compact orientable Riemannian manifold M into an open upper hemisphere S_+^{v-1} is constant.*

PROOF. As ϕ is p -harmonic

$$\operatorname{div}(\|d\Phi\|^{p-2}\nabla\Phi^\alpha) + \|d\Phi\|^p\Phi^\alpha = 0 \quad (1 \leq \alpha \leq v)$$

(the p -harmonic map system, cf. e.g. [1]) where $\Phi = \iota \circ \phi$ and $\iota: S^{v-1} \rightarrow \mathbf{R}^v$ is the inclusion. Integration over M leads (by Green's lemma) to $\int_M \|d\Phi\|^p \Phi^\alpha d\operatorname{vol}(g) = 0$ hence (by $\Phi^v > 0$) $\|d\Phi\| = 0$ and Φ is constant. Q.e.d.

We recall that $S_+^{v-1} \times S^1 \simeq S^1$ (a homotopy equivalence). Therefore a continuous map $\phi: M \rightarrow S_+^{v-1} \times S^1$ is null homotopic if and only if $\pi_2 \circ \phi: M \rightarrow S^1$ is null-homotopic. The homotopy classes of continuous maps $M \rightarrow S^1$ form an abelian group $\pi^1(M)$ (the Bruschlinski group of M , cf. e.g. [4], p. 48). Also (by Theorem 7.1 in [4], p. 49) there is a natural isomorphism $\pi^1(M) \approx H^1(M, \mathbf{Z})$. Then we may state the following

COROLLARY 1. *Let M be a compact orientable Riemannian manifold with $H^1(M, \mathbf{Z}) = 0$. Then any nonconstant p -harmonic map $\phi: M \rightarrow S^v$ meets Σ .*

4. p -Energy Minimizing Maps

Let M be a compact orientable n -dimensional Riemannian manifold, $n \geq 4$. By a result of B. Solomon, [13], any energy minimizing map $\phi: M \rightarrow S^v$ which omits a neighborhood of a codimension two totally geodesic submanifold $\Sigma \subset S^v$ is everywhere smooth. The main ingredient in the proof is R. Schoen & K. Uhlenbeck's Theorem IV in [11], p. 310. For the case of p -harmonic maps the corresponding regularity result was recovered by N. Nakauchi, [9]. Cf. also [8]. We establish

THEOREM 2. *Let M be a compact orientable n -dimensional Riemannian manifold and $\phi: M \rightarrow S^v$ a p -energy minimizing map which omits a neighborhood of a codimension two totally geodesic submanifold $\Sigma \subset S^v$. Then $\phi: M \rightarrow S^v$ is of class C^1 .*

PROOF. Let S be a Riemannian manifold and let $P(S, k)$ be the following statement: any p -energy minimizing map $\phi \in C^1(S^{d-1}, S)$, $[p] + 1 \leq d \leq k$ is constant. We need the following

LEMMA 4. *Let S be a Riemannian manifold. Let us assume that there is an integer $k \geq [p] + 1$ such that $P(S, k)$ holds true. Then $h[\operatorname{Sing}(\phi)] \leq n - k - 1$ for any p -energy minimizing map $\phi: M \rightarrow S$. If $n < k + 1$ then $\operatorname{Sing}(\phi) = \emptyset$.*

This is Proposition 1 in [9], p. 1052. Here h indicates the Hausdorff dimension (cf. §1). Let $\phi: M \rightarrow S^v$ be a p -energy minimizing map such that $\phi(M) \subset S^v \setminus V$ for some open set $V \subset S^v$ with $\Sigma \subset V$. It is well known that $H^1(S^{d-1}, \mathbf{Z}) = 0$ for any integer $d \geq 3$. By Corollary 1 above any nonconstant p -harmonic map $\phi \in C^1(S^{d-1}, S^v)$ meets Σ . Consequently $P(S^v \setminus \Sigma, k)$ holds true for any $k \geq \max\{[p] + 1, n\}$ so that (by Lemma 4) the singular set of ϕ is empty.

Only the last statement in Lemma 4 was actually used to prove Theorem 2. The following more general result may be established

THEOREM 3. *Let M and S be Riemannian manifolds where M is compact, orientable and n -dimensional. Let $p \geq 2$ and let us assume that i) the Riemannian universal covering manifold of S is a warped product manifold $\tilde{S} = L \times_w \mathbf{R}$ with the warping factor $w \in C^\infty(L)$ and ii) there is an integer $k \geq [p] + 1$ such that $P(L, k)$ holds true. Then $h[\text{Sing}(\phi)] \leq n - k - 1$ for any p -energy minimizing map $\phi \in W_{\text{loc}}^{1,p}(M, S)$.*

PROOF. As $\pi_1(S^{d-1}) = 0$ for each $d \geq 3$, any p -harmonic map $\psi \in C^1(S^{d-1}, S)$ lifts to a map $\tilde{\psi} \in C^1(S^{d-1}, \tilde{S})$ which is p -harmonic (by Lemma 2) hence factors through a p -harmonic map $\tilde{\psi} \in C^1(M, L)$ (by Lemma 1). Thus $P(L, k)$ implies $P(S, k)$ and we may apply Lemma 4. Q.e.d.

References

- [1] J. Eells & L. Lemaire, Another report on harmonic maps, Bull. London Math. Soc., (5) **20** (1988), 385–524.
- [2] A. Fardoun, Stability for the p -energy of the equator map of the ball into an ellipsoid, Diff. Geometry Appl., **8** (1998), 171–176.
- [3] A. Fardoun & R. Regbaoui, Équation de la chaleur pour les applications p -harmoniques entre variétés riemanniennes compactes, C.R. Acad. Sci. Paris, **333** (2001), 979–984.
- [4] S-T. Hu, Homology theory, Academic Press, New York-San Francisco-London, 1959.
- [5] J. Liu, Liouville-type theorems of p -harmonic maps with free boundary values, Hiroshima Math. J., (3) **40** (2010), 333–342.
- [6] A-M. Matei, Gap phenomena for p -harmonic maps, Annals of Global Analysis and Geometry, **18** (2000), 541–554.
- [7] M. Misawa, Approximation of p -harmonic maps by the penalized equation, Nonlinear Analysis, **47** (2001), 1069–1080.
- [8] N. Nakauchi, Regularity of minimizing p -harmonic maps into the sphere, Bol. U.M.I. A, (7) **10** (1996), 319–332.
- [9] N. Nakauchi, Regularity of minimizing p -harmonic maps into the sphere, Nonlinear Analysis, **47** (2001), 1051–1057.
- [10] S. Pigola & G. Veronelli, On the homotopy class of maps with finite p -energy into non-positively curved manifolds, Geom. Dedicata, **143** (2009), 109–116.
- [11] R. Schoen & K. Uhlenbeck, A regularity theory for harmonic maps, J. Differential Geometry, **19** (1984), 221–335.

- [12] R. Schoen & K. Uhlenbeck, Regularity of minimizing harmonic maps into the sphere, *Inventiones mathematicae*, **78** (1984), 89–100.
- [13] B. Solomon, Harmonic maps to spheres, *J. Differential Geometry*, **21** (1985), 151–162.
- [14] G. Veronelli, On p -harmonic maps and convex functions, *Manuscripta Math.*, (3–4) **131** (2010), 537–546.
- [15] C-Y. Wang, Minimality and perturbation of singularities for certain p -harmonic maps, *Indiana Univ. Math. J.*, (2) **47** (1998), 725–740.
- [16] S-W. Wei, Representing homotopy groups and spaces of maps by p -harmonic maps, *Indiana Univ. Math. J.*, (2) **47** (1998), 625–670.

Università degli Studi della Basilicata

Dipartimento di Matematica e Informatica

Via Dell’Ateneo Lucano 10

Contrada Macchia Romana

85100 Potenza, Italy

e-mail: sorin.dragomir@unibas.it, andrea.tommasoli@unibas.it